1. Introduction.   This is a hastily written implementation of treehull, using treaps to guarantee good average access time.

>**format** *Graph int*   /∗ *gb_graph* defines the **Graph** type and a few others ∗/
>**format** *Vertex int*
>**format** *Arc int*
>**format** *Area int*

#**include** "gb_graph.h"
#**include** "gb_miles.h"
#**include** "gb_rand.h"
>⟨ Type declarations 3 ⟩

>**int** *n* = 128;

>⟨ Global variables 2 ⟩
>⟨ Procedures 9 ⟩
>*main* ( )
>{
>   ⟨ Local variables 7 ⟩
>   **Graph** ∗*g* = *miles* (128, 0, 0, 0, 0, 0, 0);
>
>   *mems* = *ccs* = 0;
>   ⟨ Find convex hull of *g* 10 ⟩;
>   *printf* ("Total␣of␣%d␣mems␣and␣%d␣calls␣on␣ccw.\n", *mems*, *ccs*);
>}

2.   I'm instrumenting this in a simple way.

#**define** *o*   *mems* ++
#**define** *oo*   *mems* += 2
#**define** *ooo*   *mems* += 3
⟨ Global variables 2 ⟩ ≡
>   **int** *mems*;   /∗ memory accesses ∗/
>   **int** *ccs*;   /∗ calls on *ccw* ∗/

See also section 5.

This code is used in section 1.

3.   Data structures.   For now, each vertex is represented by two coordinates stored in the utility fields $x.I$ and $y.I$. I'm also putting a serial number into $z.I$, so that I can check whether different algorithms generate identical hulls.

We use separate nodes for the current convex hull. These nodes have a bunch of fields: $p\text{-}vert$ points to the vertex; $p\text{-}succ$ and $p\text{-}pred$ point to next and previous nodes in a circular list; $p\text{-}left$ and $p\text{-}right$ point to left and right children in a tree that's superimposed on the list; $p\text{-}parent$ is present too, it points to the parent node; $p\text{-}prio$ is the priority if we are implementing the tree as a treap.

The *head* node has the root of the tree in its *right* field, and it represents the special vertex that isn't in the tree.

⟨ Type declarations 3 ⟩ ≡
```
typedef struct node_struct {
    struct vertex_struct *vert;
    struct node_struct *succ, *pred, *left, *right, *parent;
    long prio;
} node;
```
This code is used in section 1.

4.   ⟨ Initialize the array of nodes 4 ⟩ ≡
```
head = (node *) gb_alloc((g-n) * sizeof(node), working_storage);
if (head ≡ Λ) return (1);      /* fixthis */
next_node = head;
```
This code is used in section 6.

5.   ⟨ Global variables 2 ⟩ +≡
```
node *head;      /* beginning of the hull data structure */
node *next_node;      /* first unused slot in that array */
Area working_storage;
int serial_no = 1;      /* used to disambiguate entries with equal coordinates */
```

6.   We assume that the vertices have been given to us in a GraphBase-type graph. The algorithm begins with a trivial hull that contains only the first two vertices.

⟨ Initialize the data structures 6 ⟩ ≡
```
init_area(working_storage);
⟨ Initialize the array of nodes 4 ⟩;
o, u = g-vertices;
v = u + 1;
u-z.I = 0;
v-z.I = 1;
p = ++next_node;
ooo, head-succ = head-pred = head-right = p;
oo, p-succ = p-pred = head;
o, p-parent = head;
oo, p-left = p-right = Λ;
gb_init_rand(110);
o, p-prio = gb_next_rand();
o, head-vert = u;
o, p-vert = v;
next_node ++;
if (n < 150) printf("Beginning with (%s; %s)\n", u-name, v-name);
```
This code is used in section 10.

7.   We'll probably need a bunch of local variables to do elementary operations on data structures.

⟨ Local variables 7 ⟩ ≡
   Vertex $*u, *v, *vv, *w$;
   node $*p, *pp, *q, *qq, *qqq, *r, *rr, *s, *ss, *tt, **par, **ppar, *prepar, *preppar$;
   int *replaced*;   /∗ will be nonzero if we've just replaced a hull element ∗/
This code is used in section 1.

8.   Here's a routine I used when debugging (in fact I should have written it sooner than I did).

⟨ Verify the integrity of the data structures 8 ⟩ ≡
  $p = head$;
  $count = 0$;
  do {
    $count \mathbin{+\!+}$;
    $p \rightarrow prio = (p \rightarrow prio \mathbin{\&} {}^{\#}\texttt{ffff0000}) + count$;
    if $(p \rightarrow succ \rightarrow pred \neq p)$ *printf* ("succ/pred␣failure␣at␣%s!\n", $p \rightarrow vert \rightarrow name$);
    if $(p \rightarrow left \neq \Lambda \wedge p \rightarrow left \rightarrow parent \neq p)$ *printf* ("parent/lchild␣failure␣at␣%s!\n", $p \rightarrow vert \rightarrow name$);
    if $(p \rightarrow right \neq \Lambda \wedge p \rightarrow right \rightarrow parent \neq p)$ *printf* ("parent/rchild␣failure␣at␣%s!\n", $p \rightarrow vert \rightarrow name$);
    $p = p \rightarrow succ$;
  } **while** $(p \neq head)$;
  $count = 1$;
  inorder ($head \rightarrow right$);
This code is used in section 10.

9.   ⟨ Procedures 9 ⟩ ≡
  int *count*;
  inorder (*p*)
      node $*p$;
  {
    if (p) {
      inorder ($p \rightarrow left$);
      if $((p \rightarrow prio \mathbin{\&} {}^{\#}\texttt{ffff}) \neq {+\!+}count)$ {
        printf ("tree␣node␣%d␣is␣missing␣at␣%d:␣%s!\n", $count, p \rightarrow prio \mathbin{\&} {}^{\#}\texttt{ffff}, p \rightarrow vert \rightarrow name$);
        $count = p \rightarrow prio \mathbin{\&} {}^{\#}\texttt{ffff}$;
      }
      inorder ($p \rightarrow right$);
    }
  }
See also sections 14, 16, and 19.
This code is used in section 1.

10.   Hull updating.   The main loop of the algorithm updates the data structure incrementally by adding one new vertex at a time. If the new vertex lies outside the current convex hull, we put it into the cycle and possibly delete some vertices that were previously part of the hull.

⟨ Find convex hull of g 10 ⟩ ≡
  ⟨ Initialize the data structures 6 ⟩;
  for $(oo, vv = g{\rightarrow}vertices + 2;\ vv < g{\rightarrow}vertices + g{\rightarrow}n;\ vv{+}{+})$ {
    $vv{\rightarrow}z.I = {+}{+}serial\_no$;
    $o, q = head{\rightarrow}pred$;
    $replaced = 0$;
    $o, u = head{\rightarrow}vert$;
    if $(o, ccw(vv, u, q{\rightarrow}vert))$ ⟨ Do Case 1 12 ⟩
    else ⟨ Do Case 2 17 ⟩;
    ⟨ Verify the integrity of the data structures 8 ⟩;
  }
  ⟨ Print the convex hull 11 ⟩;
This code is used in section 1.

11.   Let me do the easy part first, since it's bedtime and I can worry about the rest tomorrow.

⟨ Print the convex hull 11 ⟩ ≡
  $p = head$;
  $printf(\texttt{"The}_{␣}\texttt{convex}_{␣}\texttt{hull}_{␣}\texttt{is:\textbackslash n"})$;
  do {
    $printf(\texttt{"}_{␣␣}\texttt{\%s\textbackslash n"}, p{\rightarrow}vert{\rightarrow}name)$;
    $p = p{\rightarrow}succ$;
  } **while** $(p \neq head)$;
This code is used in section 10.

12.   In Case 1 we don't need the tree structure since we've already found that the new vertex is outside the hull at the tree root position.

⟨ Do Case 1 12 ⟩ ≡
  { *qqq* = *head*;
    while (1) {
       o, *r* = *qqq*→*succ*;
       if (*r* ≡ *q*) **break**;   /∗ can't eliminate any more ∗/
       if (*oo*, *ccw*(*vv*, *qqq*→*vert*, *r*→*vert*)) **break**;
       ⟨ Delete or replace *qqq* from the hull 15 ⟩;
       qqq = *r*;
    }
    qq = *qqq*;
    qqq = *q*;
    while (1) {
       o, *r* = *qqq*→*pred*;
       if (*r* ≡ *qq*) **break**;
       if (*oo*, *ccw*(*vv*, *r*→*vert*, *qqq*→*vert*)) **break**;
       ⟨ Delete or replace *qqq* from the hull 15 ⟩;
       qqq = *r*;
    }
    q = *qqq*;
    if (¬*replaced*) ⟨ Insert *vv* at the right of the tree 13 ⟩;
    if (*n* < 150) *printf*("New␣hull␣sequence␣(%s;␣%s;␣%s)\n", *q*→*vert*→*name*, *vv*→*name*, *qq*→*vert*→*name*);
  }

This code is used in section 10.

13.   At this point *q* ≡ *head*→*pred* is the tree's rightmost node.

⟨ Insert *vv* at the right of the tree 13 ⟩ ≡
  {
    tt = *next_node* ++;
    o, *tt*→*vert* = *vv*;
    o, *tt*→*succ* = *head*;
    o, *tt*→*pred* = *q*;
    o, *head*→*pred* = *tt*;
    o, *q*→*succ* = *tt*;
    oo, *tt*→*left* = *tt*→*right* = Λ;
    o, *tt*→*prio* = *gb_next_rand*( );
    if (*n* < 150) *printf*("(Inserting␣%s␣at␣right␣of␣tree,␣prio=%d)\n", *vv*→*name*, *tt*→*prio*);
    if (*o*, *tt*→*prio* < *q*→*prio*) *rotup*(*q*, &(*q*→*right*), *tt*, *tt*→*prio*);
    else {   /∗ easy case, no rotation necessary ∗/
       o, *tt*→*parent* = *q*;
       o, *q*→*right* = *tt*;
    }
  }

This code is used in section 12.

14.   The link from parent to child hasn't been set when the priorities indicate necessary rotation.

⟨ Procedures 9 ⟩ +≡

```
rotup(p, pp, q, qp)
     node *p;       /* parent of inserted node */
     node **pp;      /* link field in parent */
     node *q;       /* inserted node */
     long qp;       /* its priority */
{ node *pr, **ppr;     /* grandparent */
  node *qq;      /* child who is reparented */

  while (1) {
    o, pr = p→parent;
    if (o, pr→right ≡ p)  ppr = &(pr→right);
    else  ppr = &(pr→left);
    if (pp ≡ &(p→right)) {      /* we should rotate left */
      if (n < 150)  printf("...(rotating␣left)\n");
      o, qq = q→left;
      o, q→left = p;
      o, p→parent = q;
      o, p→right = qq;
      if (qq ≠ Λ)  o, qq→parent = p;
    }
    else {      /* we should rotate right */
      if (n < 150)  printf("...(rotating␣right)\n");
      o, qq = q→right;
      o, q→right = p;
      o, p→parent = q;
      o, p→left = qq;
      if (qq ≠ Λ)  o, qq→parent = p;
    }
    if (o, qp ≥ pr→prio)  break;
    p = pr;
    pp = ppr;
  }
  o, q→parent = pr;
  o, *ppr = q;
}
```

15.   Nodes don't need to be recycled.

⟨ Delete or replace qqq from the hull 15 ⟩ ≡
  if (*replaced*) {
    $o, pp = qqq \rightarrow pred$;
    $o, tt = qqq \rightarrow succ$;
    $o, pp \rightarrow succ = tt$;
    $o, tt \rightarrow pred = pp$;
    $o, prepar = qqq \rightarrow parent$;
    if $(o, prepar \rightarrow right \equiv qqq)$ $par = \&(prepar \rightarrow right)$;
    else $par = \&(prepar \rightarrow left)$;
    $o, pp = qqq \rightarrow left$;
    if $(o, (ss = qqq \rightarrow right) \equiv \Lambda)$ {
      if $(n < 150)$ $printf$ (`"(Deleting␣%s␣from␣tree,␣case␣1)\n"`, $qqq \rightarrow vert \rightarrow name$);
      $o, *par = pp$;
      if $(pp \neq \Lambda)$ $o, pp \rightarrow parent = prepar$;
    }
    else if $(pp \equiv \Lambda)$ {
      if $(n < 150)$ $printf$ (`"(Deleting␣%s␣from␣tree,␣case␣2)\n"`, $qqq \rightarrow vert \rightarrow name$);
      $o, *par = ss$;
      $o, ss \rightarrow parent = prepar$;
    }
    else {
      if $(n < 150)$ $printf$ (`"(Deleting␣%s␣from␣tree,␣hard␣case)\n"`, $qqq \rightarrow vert \rightarrow name$);
      $oo, deldown (prepar, par, pp, ss, pp \rightarrow prio, ss \rightarrow prio)$;
    }
  }
  else {
    $o, qqq \rightarrow vert = vv$;
    $replaced = 1$;
  }
This code is used in sections 12 and 17.

16. ⟨Procedures 9⟩ +≡

$deldown(p, pp, ql, qr, qlp, qrp)$
   node $*p$; /∗ parent of deleted node ∗/
   node $**pp$; /∗ link field in that parent ∗/
   node $*ql, *qr$; /∗ children of deleted node ∗/
   int qlp, qrp; /∗ their priorities ∗/
 { **node** $*qq$; /∗ grandchild of deleted node ∗/

  if $(qlp < qrp)$ {
   if $(n < 150)$ $printf($"...(moving␣left␣child␣up)\n"$)$;
   $o, ql{\rightarrow}parent = p$;
   $o, *pp = ql$;
   $o, qq = ql{\rightarrow}right$;
   if $(qq \neq \Lambda)$ $o, deldown(ql, \&(ql{\rightarrow}right), qq, qr, qq{\rightarrow}prio, qrp)$; /∗ tail recursion ∗/
   else {
    $o, ql{\rightarrow}right = qr$;
    $o, qr{\rightarrow}parent = ql$;
   }
  }
  else {
   if $(n < 150)$ $printf($"...(moving␣right␣child␣up)\n"$)$;
   $o, qr{\rightarrow}parent = p$;
   $o, *pp = qr$;
   $o, qq = qr{\rightarrow}left$;
   if $(qq \neq \Lambda)$ $o, deldown(qr, \&(qr{\rightarrow}left), ql, qq, qlp, qq{\rightarrow}prio)$; /∗ tail recursion ∗/
   else {
    $o, qr{\rightarrow}left = ql$;
    $o, ql{\rightarrow}parent = qr$;
   }
  }
 }

17. ⟨Do Case 2 17⟩ ≡

  { $o, qq = head \rightarrow right$;

    while (1) {

      if $(qq \equiv q \vee (o, ccw(u, vv, qq \rightarrow vert)))$ {

        $o, r = qq \rightarrow left$;

        if $(r \equiv \Lambda)$ {

          $preppar = qq$;

          $o, ppar = \&(qq \rightarrow left)$;

          break;

        }

      }

      else {

        $o, r = qq \rightarrow right$;

        if $(r \equiv \Lambda)$ {

          $preppar = qq$;

          $o, ppar = \&(qq \rightarrow right)$;

          $o, qq = qq \rightarrow succ$;

          break;

        }

      }

      $qq = r$;

    }

    if $(o, (r = qq \rightarrow pred) \equiv head \vee (oo, ccw(vv, qq \rightarrow vert, r \rightarrow vert)))$ {

      if $(r \neq head)$ {

        while (1) {

          $qqq = r$;

          $o, r = qqq \rightarrow pred$;

          if $(r \equiv head)$ **break**;

          if $(oo, ccw(vv, r \rightarrow vert, qqq \rightarrow vert))$ **break**;

          ⟨Delete or replace qqq from the hull 15⟩;

        }

        $r = qqq$;

      }

      $qqq = qq$;

      while (1) {

        if $(qqq \equiv q)$ **break**;

        $oo, rr = qqq \rightarrow succ$;

        if $(oo, ccw(vv, qqq \rightarrow vert, rr \rightarrow vert))$ **break**;

        ⟨Delete or replace qqq from the hull 15⟩;

        $qqq = rr$;

      }

      if $(\neg replaced)$ ⟨Insert vv in tree, linked by $ppar$ 18⟩;

      if $(n < 150)$

        $printf(\texttt{"New\_hull\_sequence\_(\%s;\_\%s;\_\%s)\textbackslash n"}, r \rightarrow vert \rightarrow name, vv \rightarrow name, qqq \rightarrow vert \rightarrow name)$;

    }

  }

This code is used in section 10.

18. ⟨Insert *vv* in tree, linked by *ppar* 18⟩ ≡

  {

    $tt = next\_node \mathbin{+}\mathbin{+}$;

    $o, tt\rightarrow vert = vv$;

    $o, tt\rightarrow succ = qq$;

    $o, tt\rightarrow pred = r$;

    $o, qq\rightarrow pred = tt$;

    $o, r\rightarrow succ = tt$;

    $oo, tt\rightarrow left = tt\rightarrow right = \Lambda$;

    $o, tt\rightarrow prio = gb\_next\_rand(\,)$;

    if $(n < 150)$ $printf(\texttt{"(Inserting}_{\sqcup}\texttt{\%s}_{\sqcup}\texttt{at}_{\sqcup}\texttt{bottom}_{\sqcup}\texttt{of}_{\sqcup}\texttt{tree,}_{\sqcup}\texttt{prio=\%d)\textbackslash n"}, vv\rightarrow name, tt\rightarrow prio)$;

    if $(o, tt\rightarrow prio < preppar\rightarrow prio)$ $rotup(preppar, ppar, tt, tt\rightarrow prio)$;

    else {   /∗ easy case, no rotation needed ∗/

      $o, tt\rightarrow parent = preppar$;

      $o, *ppar = tt$;

    }

  }

This code is used in section 17.

19. Determinants.   I need code for the primitive function *ccw*. Floating-point arithmetic suffices for my purposes.

We want to evaluate the determinant

$$ccw(u, v, w) = \begin{vmatrix} u(x) & u(y) & 1 \\ v(x) & v(y) & 1 \\ w(x) & w(y) & 1 \end{vmatrix} = \begin{vmatrix} u(x) - w(x) & u(y) - w(y) \\ v(x) - w(x) & v(y) - w(y) \end{vmatrix} .$$

⟨ Procedures 9 ⟩ +≡
  int *ccw* (*u, v, w*)
      Vertex *∗u, ∗v, ∗w*;
  { **register double** *wx* = (**double**) *w⃗x.I, wy* = (**double**) *w⃗y.I*;
    register double *det* = ((**double**) *u⃗x.I* − *wx*) ∗ ((**double**) *v⃗y.I* − *wy*) − ((**double**) *u⃗y.I* − *wy*) ∗
      ((**double**) *v⃗x.I* − *wx*);
    Vertex *∗uu* = *u, ∗vv* = *v, ∗ww* = *w, ∗t*;
    if (*det* ≡ 0) {
      det = 1;
      if (*u⃗x.I* > *v⃗x.I* ∨ (*u⃗x.I* ≡ *v⃗x.I* ∧ (*u⃗y.I* > *v⃗y.I* ∨ (*u⃗y.I* ≡ *v⃗y.I* ∧ *u⃗z.I* > *v⃗z.I*)))) {
        t = *u*; *u* = *v*; *v* = *t*; *det* = −*det*;
      }
      if (*v⃗x.I* > *w⃗x.I* ∨ (*v⃗x.I* ≡ *w⃗x.I* ∧ (*v⃗y.I* > *w⃗y.I* ∨ (*v⃗y.I* ≡ *w⃗y.I* ∧ *v⃗z.I* > *w⃗z.I*)))) {
        t = *v*; *v* = *w*; *w* = *t*; *det* = −*det*;
      }
      if (*u⃗x.I* > *v⃗x.I* ∨ (*u⃗x.I* ≡ *v⃗x.I* ∧ (*u⃗y.I* > *v⃗y.I* ∨ (*u⃗y.I* ≡ *v⃗y.I* ∧ *u⃗z.I* < *v⃗z.I*)))) {
        det = −*det*;
      }
    }
    if (*n* < 150)
      printf ("cc(%s;␣%s;␣%s)␣is␣%s\n", *uu⃗name, vv⃗name, ww⃗name, det* > 0 ? "true" : "false");
    ccs ++;
    return (*det* > 0);
  }

prio: 3, 6, 8, 9, 13, 14, 15, 16, 18.
q: 7, 14.
ql: 16.
qlp: 16.
qp: 14.
qq: 7, 12, 14, 16, 17, 18.
qqq: 7, 12, 15, 17.
qr: 16.
qrp: 16.
r: 7.
replaced: 7, 10, 12, 15, 17.
right: 3, 6, 8, 9, 13, 14, 15, 16, 17, 18.
rotup: 13, 14, 18.
rr: 7, 17.
s: 7.
serial_no: 5, 10.
ss: 7, 15.
succ: 3, 6, 8, 11, 12, 13, 15, 17, 18.
t: 19.
tt: 7, 13, 15, 18.
u: 7, 19.
uu: 19.
v: 7, 19.
vert: 3, 6, 8, 9, 10, 11, 12, 13, 15, 17, 18.
Vertex: 7, 19.
vertex_struct: 3.
vertices: 6, 10.
vv: 7, 10, 12, 13, 15, 17, 18, 19.
w: 7, 19.
working_storage: 4, 5, 6.
ww: 19.
wx: 19.
wy: 19.

⟨ Delete or replace *qqq* from the hull 15 ⟩   Used in sections 12 and 17.
⟨ Do Case 1 12 ⟩   Used in section 10.
⟨ Do Case 2 17 ⟩   Used in section 10.
⟨ Find convex hull of *g* 10 ⟩   Used in section 1.
⟨ Global variables 2, 5 ⟩   Used in section 1.
⟨ Initialize the array of nodes 4 ⟩   Used in section 6.
⟨ Initialize the data structures 6 ⟩   Used in section 10.
⟨ Insert *vv* at the right of the tree 13 ⟩   Used in section 12.
⟨ Insert *vv* in tree, linked by *ppar* 18 ⟩   Used in section 17.
⟨ Local variables 7 ⟩   Used in section 1.
⟨ Print the convex hull 11 ⟩   Used in section 10.
⟨ Procedures 9, 14, 16, 19 ⟩   Used in section 1.
⟨ Type declarations 3 ⟩   Used in section 1.
⟨ Verify the integrity of the data structures 8 ⟩   Used in section 10.